Directed Cyclic Graph for Causal Discovery from Multivariate Functional Data

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Joint work with

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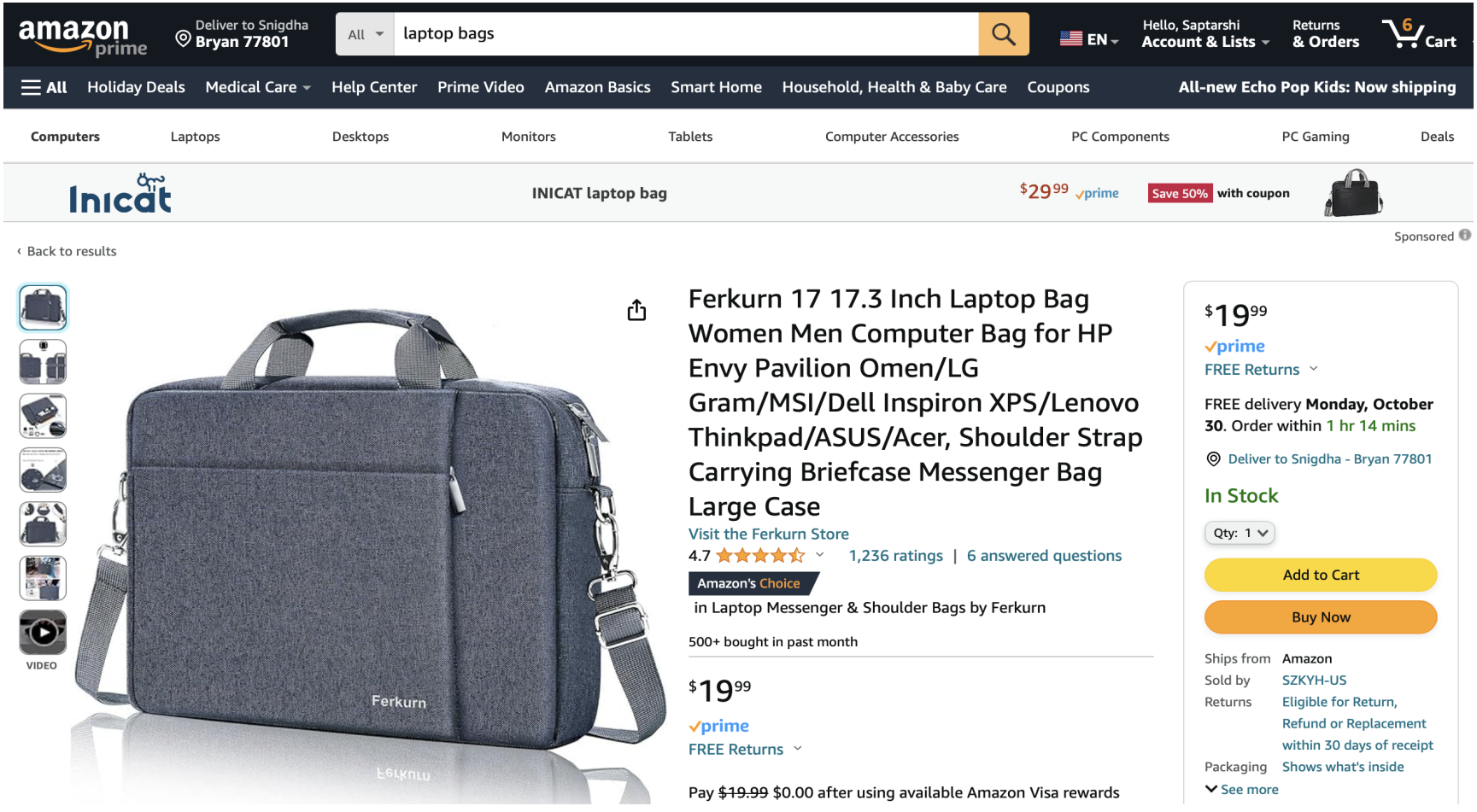
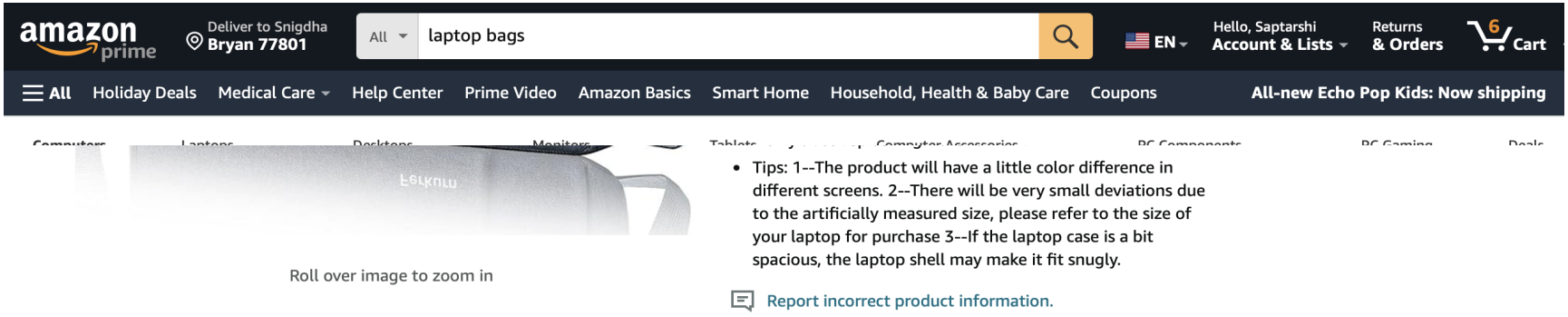
Why causality?

Example 1



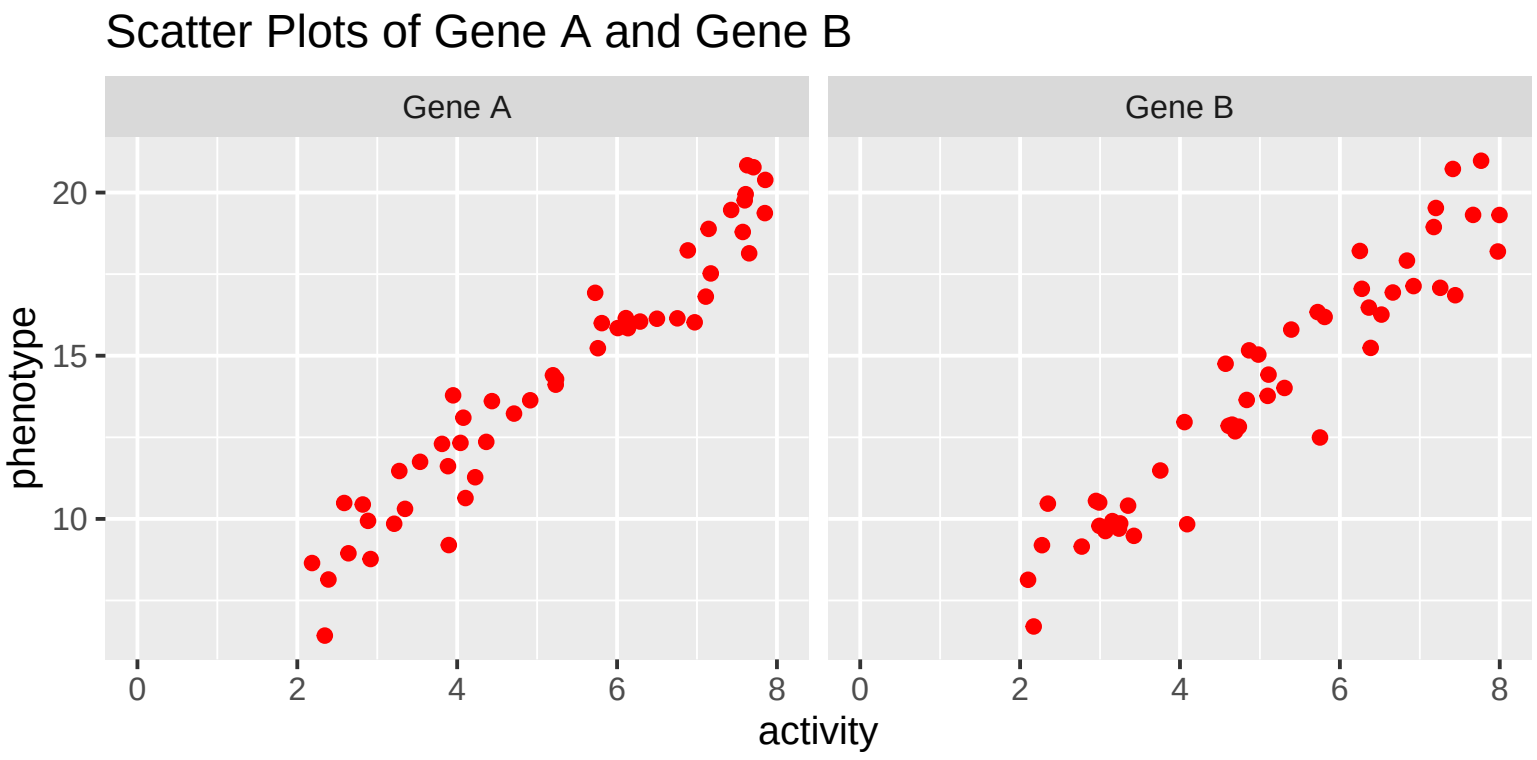
Why causality?

Example 1



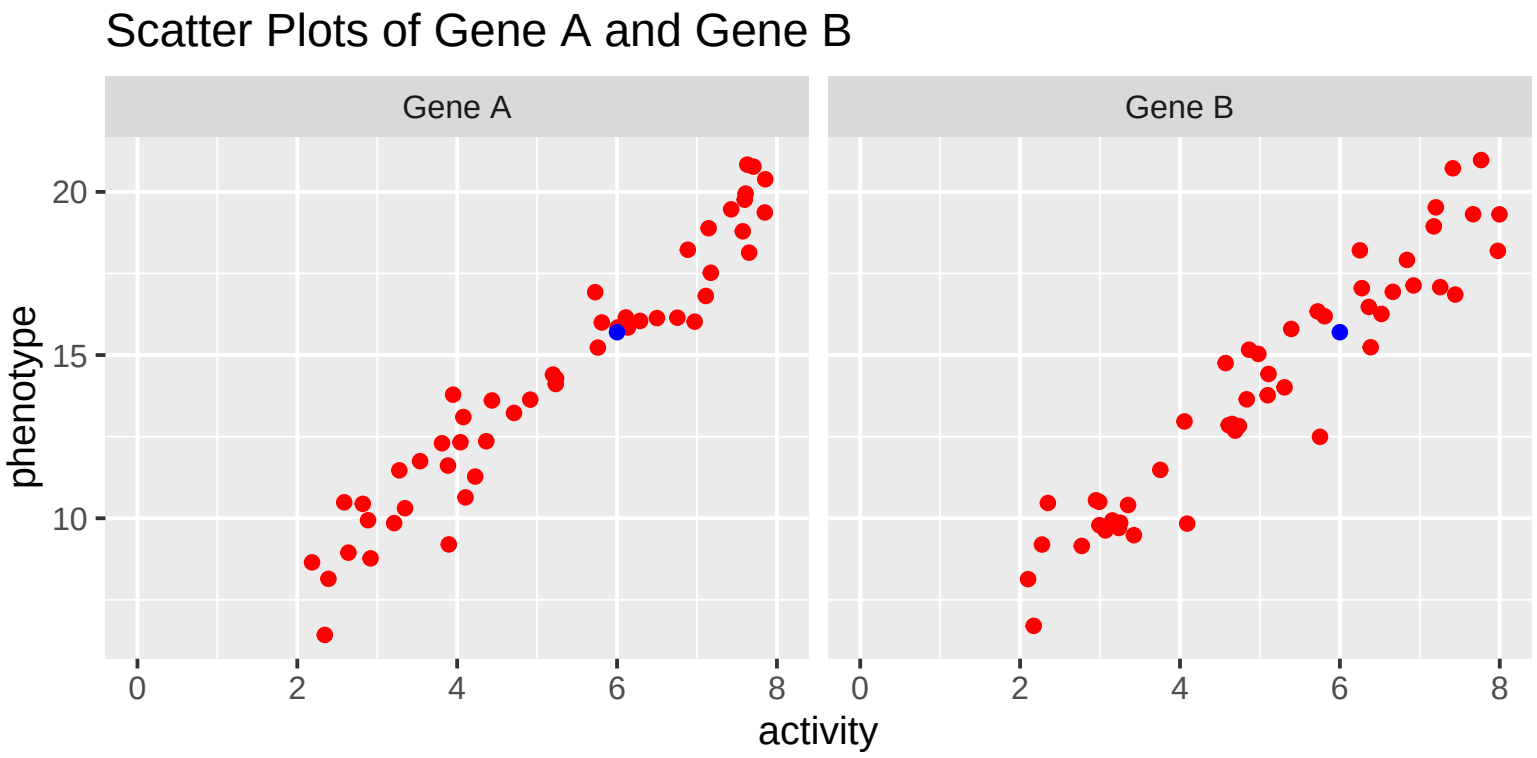
Why causality?

Example 2



Why causality?

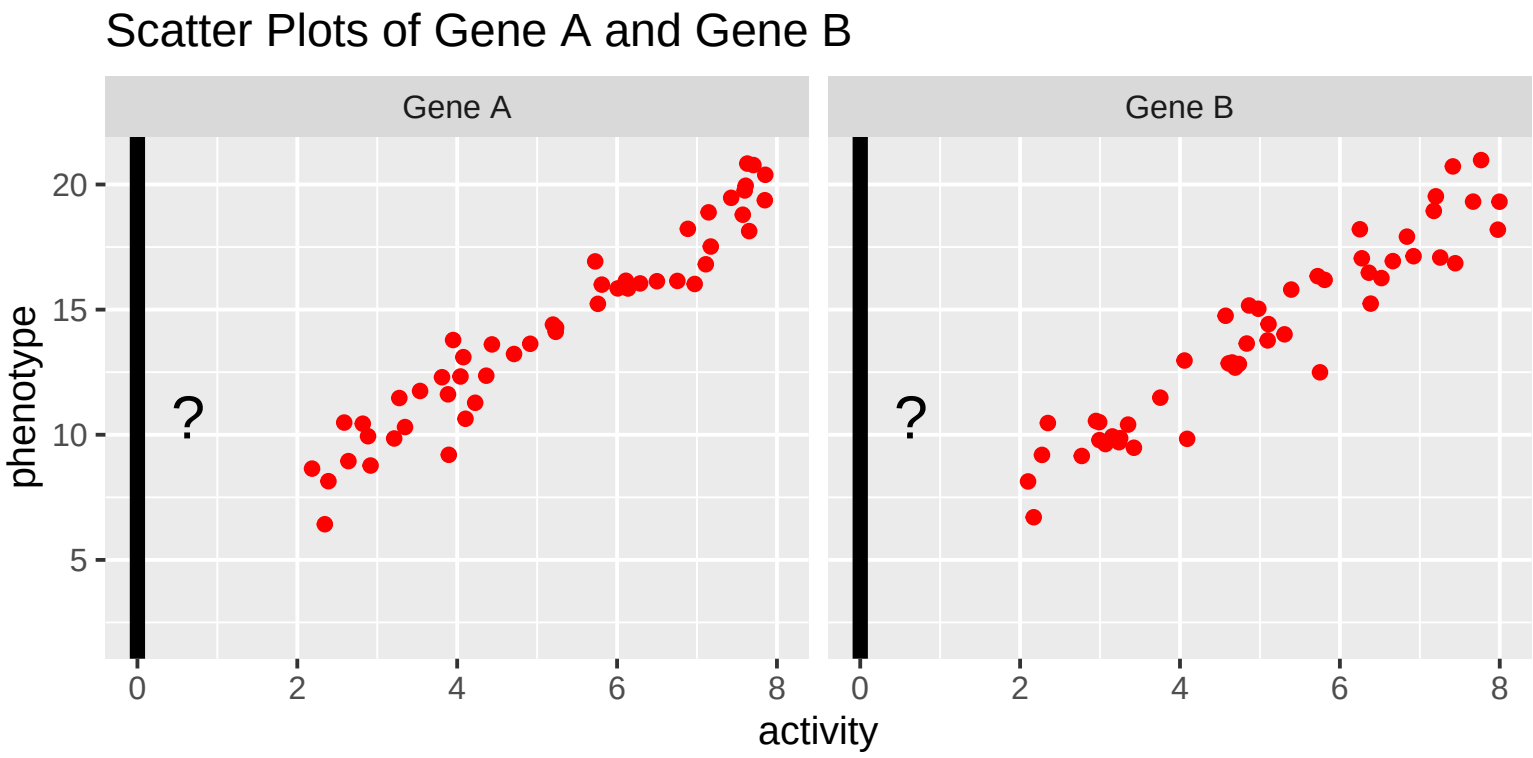
Example 2



Statistical Question!!

Why causality?

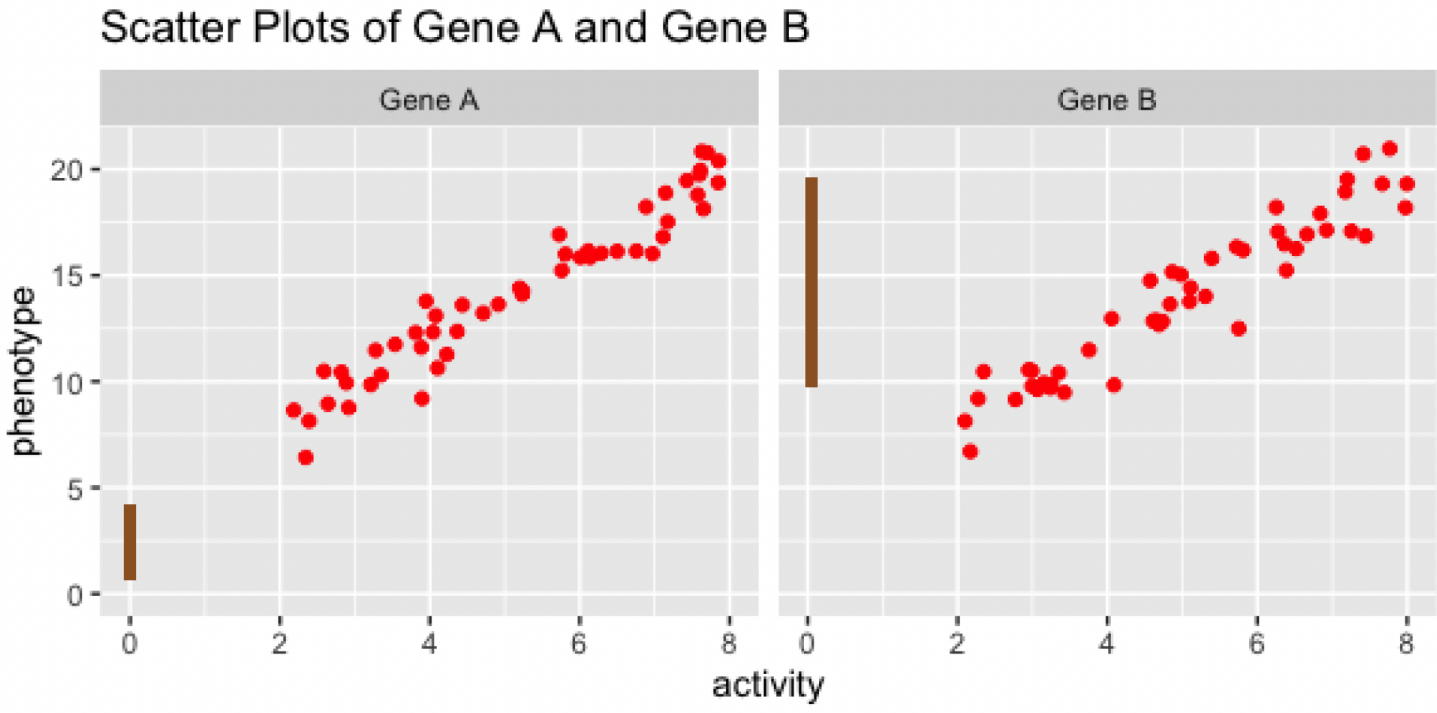
Example 2



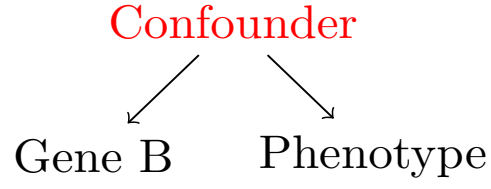
Causal Question!!

Causality matters

Example 2



Gene A $\longrightarrow$ Phenotype



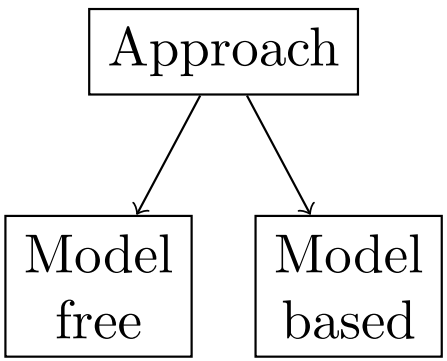
Causal discovery

- Causal discovery - infer causal structure from data.
- Gold standard is experimental data, however mostly we have observational data at our disposal.
- Fairly a difficult task to do.
- Suppose $X_1, \ldots, X_p$ forms a graph $\mathcal{G}$. Goal is to learn $\mathcal{G}$.
- Some usual assumptions:-
  1. Causal Markov condition – $X_i \perp\!\!\!\perp X_{\text{NonDesc}(i)} | X_{\text{pa}(i)}$.

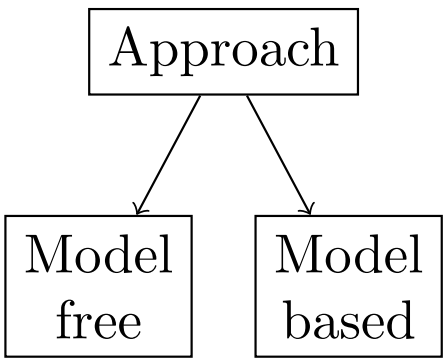
     smoking $\rightarrow$ tissue damage $\rightarrow$ lung cancer

  2. Causal faithfulness – $X_i \perp\!\!\!\perp_p X_j | X_k \implies X_i \perp\!\!\!\perp_{\mathcal{G}} X_j | X_k$.
  3. Causal sufficiency – Absence of any hidden confounders.
  4. Acyclicity – Use of Directed Acyclic Graphs (DAGs).

Current approaches

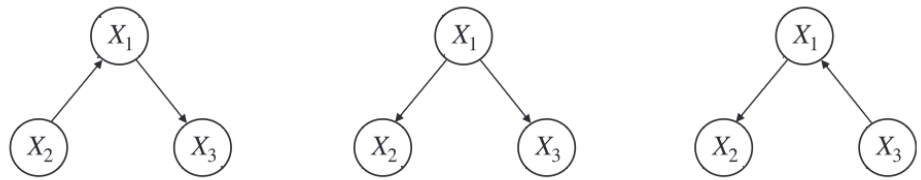


Current approaches



Model free approach:

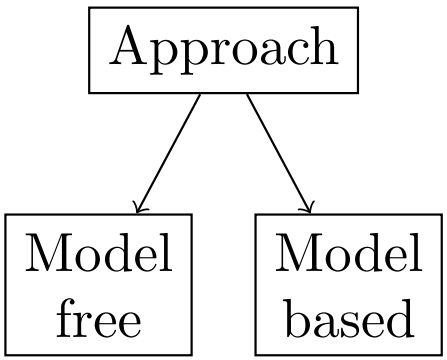
- Uses conditional independence tests – $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y} | \boldsymbol{Z}$.
- Non-parametric approach.
- Assumptions: causal Markov condition, causal faithfulness.
- determines the orientations of the edges up to the Markov equivalence class.
- Examples: PC, FCI, CCD.



Markov equivalence class of trivariate graphs - $X_2 \perp\!\!\!\perp X_3 | X_1$

Question: Can we do better than this?

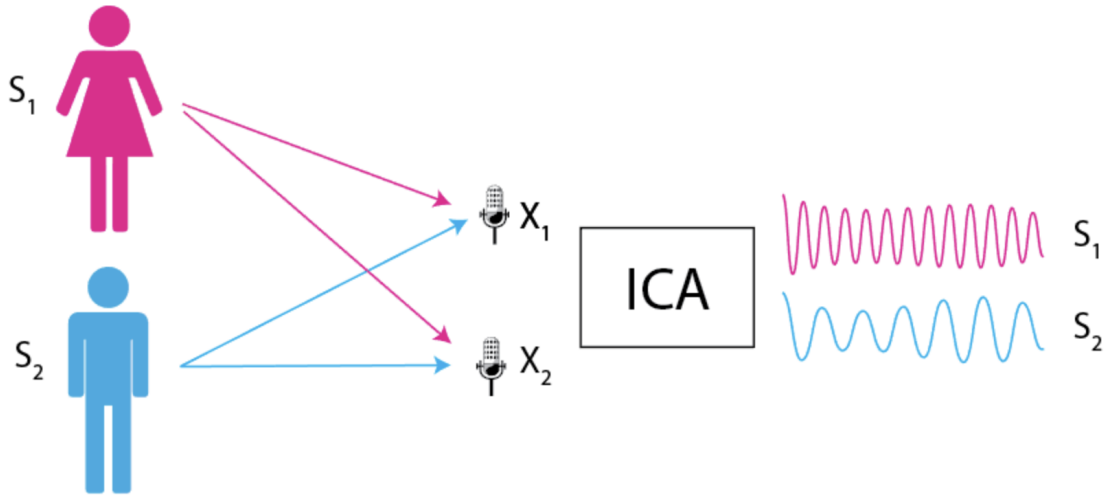
Current approaches



Model based approach:

- Structural equation model (SEM) $X_i = f_i(\boldsymbol{X}_{\mathrm{pa}(i)}, \epsilon_i)$.
- Parametric approach.
- Restricts the function class.
- The causal model is fully identifiable.
- Examples - LiNGAM, LiNG, PNL, Non-linear Additive Noise models etc.
- $f(\boldsymbol{X}_{\mathrm{pa}(i)}, \epsilon_i) = \boldsymbol{b}_i^\top \boldsymbol{X}_{\mathrm{pa}(i)} + \epsilon_i$

Independent component analysis



Goal is to recover the independent source signals S_i.

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S}$$

ICA & LiNGAM

ICA

- Model is $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S}$.
- Goal: to estimate $\boldsymbol{W} = \boldsymbol{A}^{-1}$.

LiNGAM

- Model is $(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{X} = \boldsymbol{E}$.
- Underlying graph structure is acyclic.
- Goal: to estimate $\boldsymbol{B}$.

ICA & LiNGAM

ICA

- Model is $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S}$.
- Goal: to estimate $\boldsymbol{W} = \boldsymbol{A}^{-1}$.
- S_i are non-gaussian and mutually independent.
- Identifiable upto scaling and permutation indeterminacy of indep. components $\boldsymbol{S}$.

$$\boldsymbol{X} = \underbrace{(\boldsymbol{A}\boldsymbol{P}\boldsymbol{D})}_{\boldsymbol{A}^*}\underbrace{(\boldsymbol{D}^{-1}\boldsymbol{P}^\top\boldsymbol{S})}_{\boldsymbol{S}^*}$$

- several algorithms leads to efficient estimation of $\boldsymbol{W}$.

LiNGAM

- Model is $(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{X} = \boldsymbol{E}$.
- Underlying graph structure is acyclic.
- Goal: to estimate $\boldsymbol{B}$.

ICA & LiNGAM

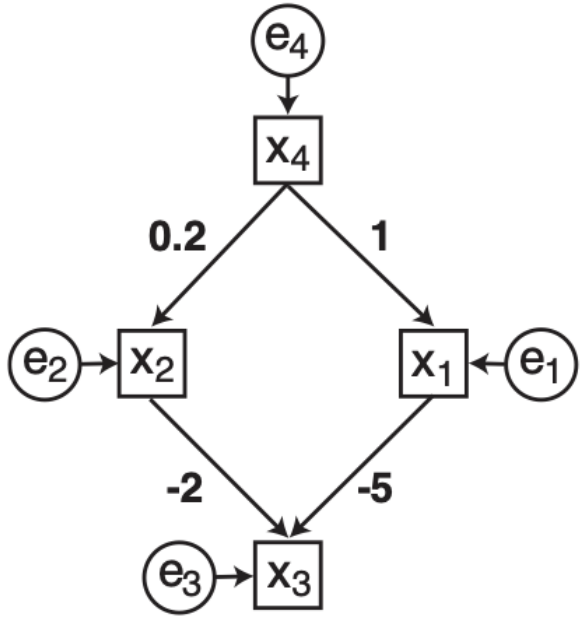
ICA

- Model is $\boldsymbol{X} = \boldsymbol{AS}$.
- Goal: to estimate $\boldsymbol{W} = \boldsymbol{A}^{-1}$.
- S_i are non-gaussian and mutually independent.
- Identifiable upto scaling and permutation indeterminacy of indep. components $\boldsymbol{S}$.

$$\boldsymbol{X} = \underbrace{(\boldsymbol{APD})}_{\boldsymbol{A}^*}\underbrace{(\boldsymbol{D}^{-1}\boldsymbol{P}^\top\boldsymbol{S})}_{\boldsymbol{S}^*}$$

- several algorithms leads to efficient estimation of $\boldsymbol{W}$.

LiNGAM



Unlike ICA, the correct correspondence between e_i and x_i is important in LiNGAM.

ICA & LiNGAM

ICA

- Model is $\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S}$.
- Goal: to estimate $\boldsymbol{W} = \boldsymbol{A}^{-1}$.
- S_i are non-gaussian and mutually independent.
- Identifiable upto scaling and permutation indeterminacy of indep. components $\boldsymbol{S}$.

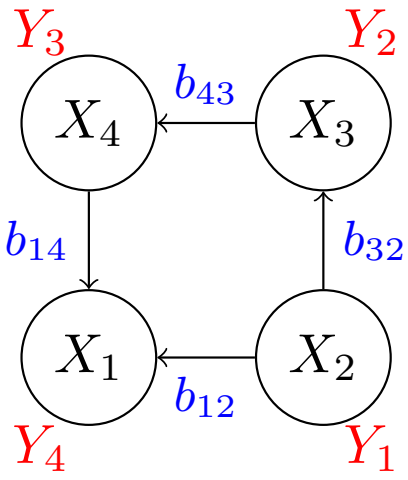
$$\boldsymbol{X} = \underbrace{(\boldsymbol{A}\boldsymbol{P}\boldsymbol{D})}_{\boldsymbol{A}^*}\underbrace{(\boldsymbol{D}^{-1}\boldsymbol{P}^\top\boldsymbol{S})}_{\boldsymbol{S}^*}$$

- several algorithms leads to efficient estimation of $\boldsymbol{W}$.

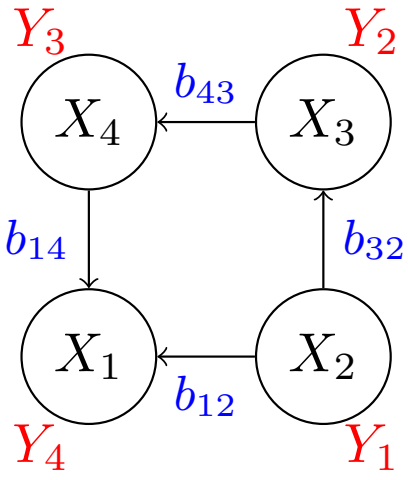
LiNGAM

- Model is $(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{X} = \boldsymbol{E}$.
- Underlying graph structure is acyclic.
- Goal: to estimate $\boldsymbol{B}$.
- E_i are non-gaussian and mutually independent.
- Some post-processing needed on the $\widehat{\boldsymbol{W}}$.
- Efficient algorithm is provided in Shimizu et al. (2006).

Presence of cycles - Difficulty?



Presence of cycles - Difficulty?

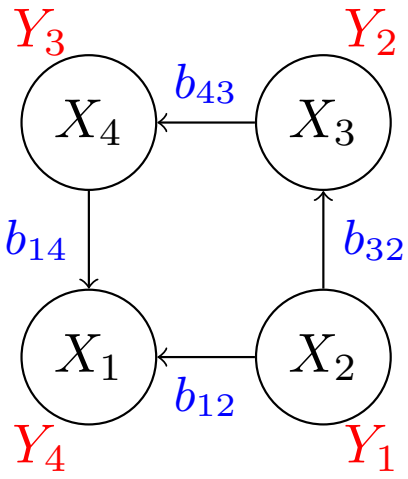


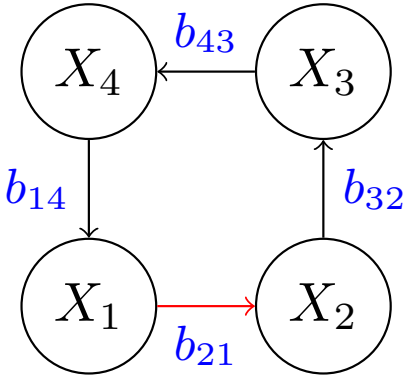
- DAGs iff topological order.
- Topological sort/order: $Y_i \to Y_j \implies j > i$.
- Example: $Y_1(X_2)$ $Y_2(X_3)$ $Y_3(X_4)$ $Y_4(X_1)$

$$Y_1 \longrightarrow Y_2 \longrightarrow Y_3 \longrightarrow Y_4$$

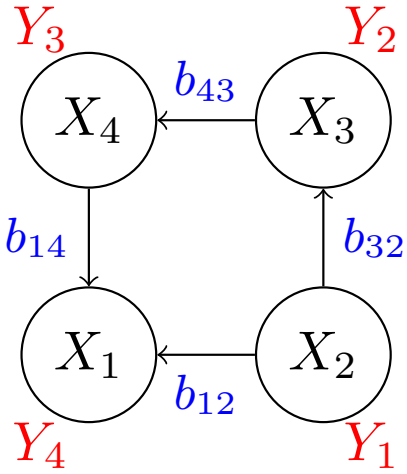
- $$\begin{pmatrix} Y_1 \\ \vdots \\ Y_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{32} & 0 & 0 & 0 \\ 0 & b_{43} & 0 & 0 \\ b_{12} & 0 & b_{14} & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_4 \end{pmatrix} + \begin{pmatrix} e_2 \\ e_3 \\ e_4 \\ e_1 \end{pmatrix}$$
- $\boldsymbol{I} - \boldsymbol{B}$ is always invertible.

Presence of cycles - Difficulty?

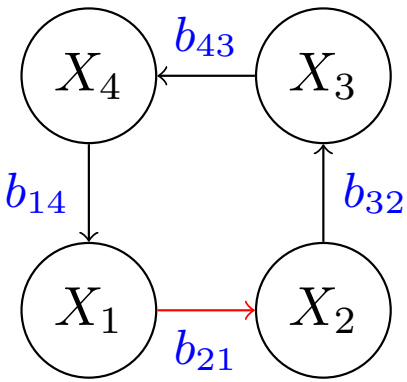




Presence of cycles - Difficulty?



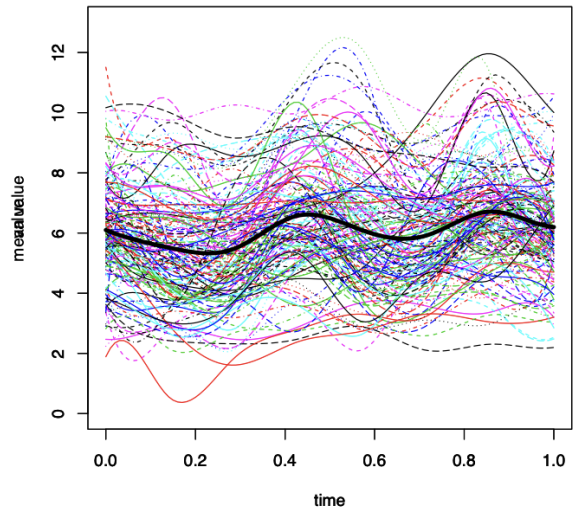
$$(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{X} = \boldsymbol{E}$$



- $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 & b_{14} \\ b_{21} & 0 & 0 & 0 \\ 0 & b_{32} & 0 & 0 \\ 0 & 0 & b_{43} & 0 \end{pmatrix}$
- $|\boldsymbol{I} - \boldsymbol{B}| = 1 - b_{14}b_{21}b_{32}b_{43}$.
- Some extra conditions are neccessary on $\boldsymbol{B}$ to ensure invertibility.
- The moduli of the eigenvalues of $\boldsymbol{B}$ are less than 1 and none equal to 1.

Definition

- Data where the data points are itself functions or curves.
- A realization of a (typically smooth) random object that takes values in an abstract function space.
- $\mathcal{H} := \{X | X : \mathcal{T} \to [a, b]\}$. Example: Hilbert space.
- Denoted by $X = (X_1, \ldots, X_p)^\top$.
- Examples: 1) daily PM10 concentration curves recorded in Graz, Austria in winter season 2) electrical activity measured across different regions of the brain (EEG data).



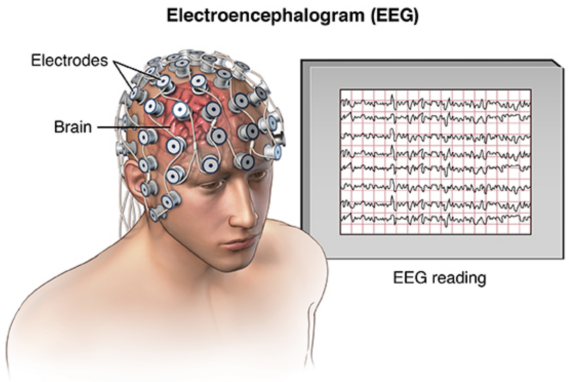
Source: Internet

Challenges

1. Infinite-dimensionality of functional data
   - Low-freq spectrum of Y_j might causally affect the high-freq. spectrum of Y_ℓ.
   - Demands identification of pertinent features.
   - The challenge is that we may not know *a priori* what these relevant features are.
2. Causal identifiability theory in presence of cycles/feedback loops.
3. Noisy functional data adds another layer of difficulty in probing the causal relationships of interest.

Motivating example

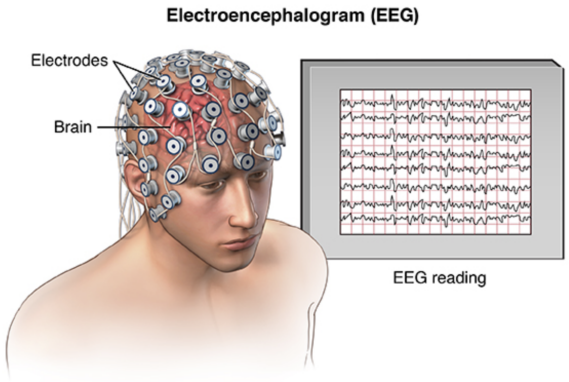
- Electroencephalography (EEG) data.



Source: Internet

Motivating example

- Electroencephalography (EEG) data.
- Continuous and the short time separation between the adjacent measuring points.
- **Goal**: To estimate brain effective connectivity among different regions.
- Strong biological evidence behind presence of feedback loops/cycles.



Source: Internet

Model definition

- Consider a multivariate stochastic process $\boldsymbol{Y} = (Y_1, \ldots, Y_p)^\top$ where each Y_j is defined on a compact domain $\mathcal{T}_j \subset \mathbb{R}$.
- Let $Y_j \in \mathcal{H}_j$ where $\mathcal{H}_j$ is a Hilbert space of functions defined on $\mathcal{T}_j$.
- Consider an operator-based non-recursive linear SEM on $\boldsymbol{Y}$ as

$$Y_j(\cdot) = \sum_{\ell \in \mathrm{pa}(j)} (\mathcal{B}_{j\ell} Y_\ell)(\cdot) + f_j(\cdot), \quad \forall j \in [p], \tag{1}$$

- $Y_\ell \to Y_j \implies \mathcal{B}_{j\ell} \neq 0$. Assume $\mathcal{B}_{jj}$ is a null operator (no self loops).
- Suppose if $\boldsymbol{Y}_j \in \mathbb{R}^m$ $\forall j$,, (1) is just,

$$\boldsymbol{Y}_j = \sum_{\ell \in \mathrm{pa}(j)} \boldsymbol{B}_{j\ell} \boldsymbol{Y}_\ell + \boldsymbol{f}_j, \quad \forall j \in [p],$$

- However, model (1) is infinite-dimensional and hence challenging to estimate and interpret.

Causal embedded space

- Assume that the causal relationships are preserved in an unknown low-dimensional subspace $\mathcal{D}_j$ of $\mathcal{H}_j$ of dim K_j.
- $\mathcal{P}_j$ and $\mathcal{Q}_j$ are the projections onto $\mathcal{D}_j$ and its orthogonal complement resp.
- Also assume $\mathcal{B}_{j\ell} = \mathcal{P}_j \mathcal{B}_{j\ell} \mathcal{P}_\ell$.
- As such, (1) can be split into

$$\mathcal{P}_j Y_j = \sum_{\ell \in \mathrm{pa}(j)} \mathcal{B}_{j\ell}(\mathcal{P}_\ell Y_\ell) + \mathcal{P}_j f_j, \tag{2}$$

$$\mathcal{Q}_j Y_j = \mathcal{Q}_j f_j.$$

- We assume that $\mathcal{P}_j f_j$ and $\mathcal{Q}_j f_j$ are independent of each other.

Model definition

- In practice, we do not directly observe Y_j.
- For each Y_j, we observe $\{(t_{ju}, X_{ju})\}_{u=1}^{m_j}$, where $X_{ju} \in \mathbb{R}$ is the measurement of Y_j at location $t_{ju} \in \mathcal{T}_j$.
- Therefore, we consider the following measurement model:

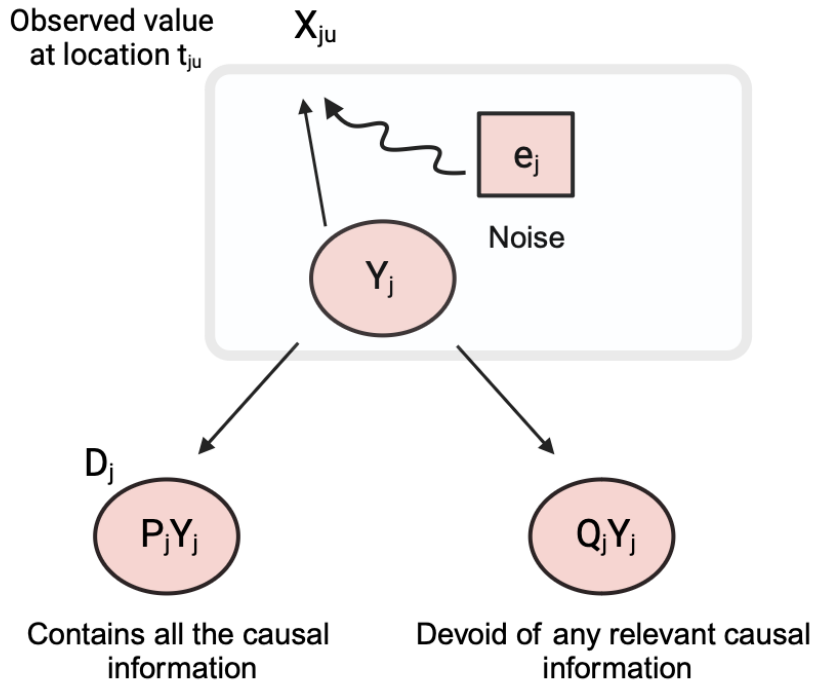
$$\begin{aligned} X_{ju} &= Y_j(t_{ju}) + e_{ju} \\ &= (\mathcal{P}_j Y_j)(t_{ju}) + (\mathcal{Q}_j Y_j)(t_{ju}) + e_{ju}, \quad \forall u \in [m_j], j \in [p], \end{aligned} \tag{3}$$

with independent noises $e_{ju} \sim N(0, \sigma_j), \forall u \in [m_j]$.

Model definition

Recall that,

$$X_{ju} = (\mathcal{P}_j Y_j)(t_{ju}) + (\mathcal{Q}_j Y_j)(t_{ju}) + e_{ju}$$



Model definition

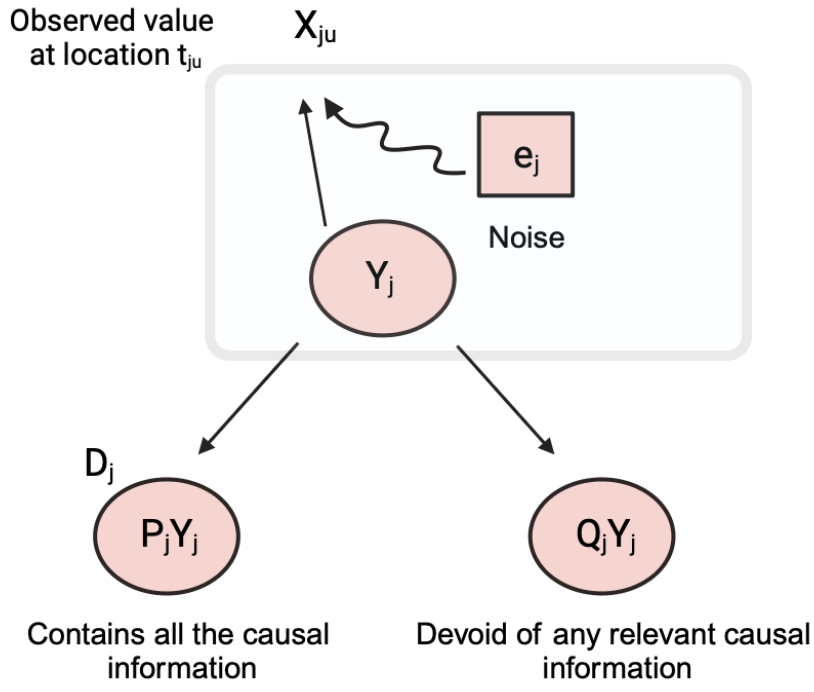
Recall that,

$$X_{ju} = (\mathcal{P}_j Y_j)(t_{ju}) + (\mathcal{Q}_j Y_j)(t_{ju}) + e_{ju}$$

- Define $\alpha_j = \mathcal{P}_j Y_j$, $\beta_j = \mathcal{Q}_j Y_j$ and $\epsilon_j = \mathcal{P}_j f_j, \forall j \in [p]$.
- More compactly,

$$\boldsymbol{X} = \boldsymbol{\alpha}(\boldsymbol{t}) + \boldsymbol{\beta}(\boldsymbol{t}) + \boldsymbol{e} \quad (4)$$

- **Question**: Is the proposed model causally identifiable? Yes, under certain assump.



Assumptions

Assumption 1 (Causal Sufficiency)

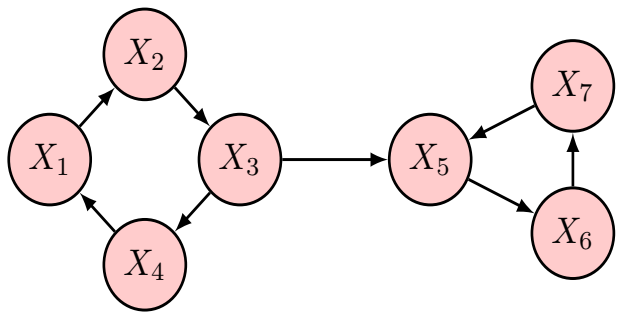
The model $\mathcal{S} = (\mathcal{G}, \mathbb{P})$ is causally sufficient, i.e., there are no unmeasured confounders.

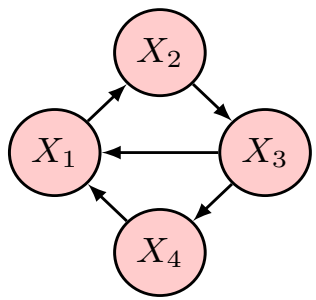
- Keeps the causal discovery task more manageable especially for cyclic graphs with purely observational data.

Assumptions contd...

Assumption 2 (Disjoint Cycles)

The cycles in $\mathcal{G}$ are disjoint, i.e., no two cycles in the graph have two nodes that are common to both.





Assumptions contd...

Assumption 3 (Stability)

For the model $\mathcal{S}$, the moduli of the eigenvalues of the finite rank operator $\mathcal{B}$ are less than or equal to 1, and none of the real eigenvalues are equal to 1.

- Similar kind of assumptions that were discussed for the univariate case have been extended for the random functions.

Assumptions contd...

Assumption 4 (Non-Gaussianity)

The exogenous variables have independent mixture of Gaussian distributions. i.e., $\epsilon_{jk} \overset{\text{ind}}{\sim} \sum_{m=1}^{M_{jk}} \pi_{jkm} \mathrm{N}(\mu_{jkm}, \tau_{jkm})$ with $M_{jk} \geq 2$.

- Can approx. any cont. distribution arbitrarily well [1]
- Useful as it induces model identifiability in the linear SEM framework.[2]

[1] Titterington, Smith, & Makov (1985). Statistical analysis of finite mixture distributions.

[2] Shimizu, Hoyer, Hyvärinen, & Kerminen. (2006). A Linear Non-Gaussian Acyclic Model for Causal Discovery.

Assumptions contd...

Recall that,

$$X_{ju} = (\mathcal{P}_j Y_j)(t_{ju}) + (\mathcal{Q}_j Y_j)(t_{ju}) + e_{ju}$$

$$\beta_j = \mathcal{Q}_j Y_j$$

Assumption 5 (Non-causal dependency)

Let $\boldsymbol{\beta}(\boldsymbol{t}) = \boldsymbol{C}(\boldsymbol{t})\boldsymbol{\gamma}$, where $\boldsymbol{C}(\boldsymbol{t}) = \text{diag}(\boldsymbol{C}_{11}(\boldsymbol{t}_1), \ldots, \boldsymbol{C}_{pp}(\boldsymbol{t}_p))$ and $\boldsymbol{\gamma}$ be another exogenous component. We assume $\gamma_{jk} \overset{\text{ind}}{\sim} \sum_{m=1}^{M_{jk}} \pi'_{jkm} \text{N}(\mu'_{jkm}, \tau'_{jkm})$ with $M_{jk} \geq 1$.

- $\boldsymbol{\beta}_j(\boldsymbol{t}_j) \perp\!\!\!\perp \boldsymbol{\beta}_\ell(\boldsymbol{t}_\ell)$ for $j \neq \ell$ and $j, \ell \in [p]$.
- $\boldsymbol{C}_{jj}(\boldsymbol{t}_j)$ mixes the independent entries in $\boldsymbol{\gamma}$ to generate temporal dependence within $\boldsymbol{\beta}_j(\boldsymbol{t}_j)$.

Assumptions contd...

Recall, $\boldsymbol{X} = \boldsymbol{\alpha}(\boldsymbol{t}) + \boldsymbol{\beta}(\boldsymbol{t}) + \boldsymbol{e}$. Then if $\alpha_j(t_{ju}) = \sum_{k=1}^{K_j} \tilde{\alpha}_{jk}\phi_{jk}(t_{ju})$, then (4) can be written as,

$$\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{t})\tilde{\boldsymbol{\alpha}} + \boldsymbol{\beta}(\boldsymbol{t}) + \boldsymbol{e},$$

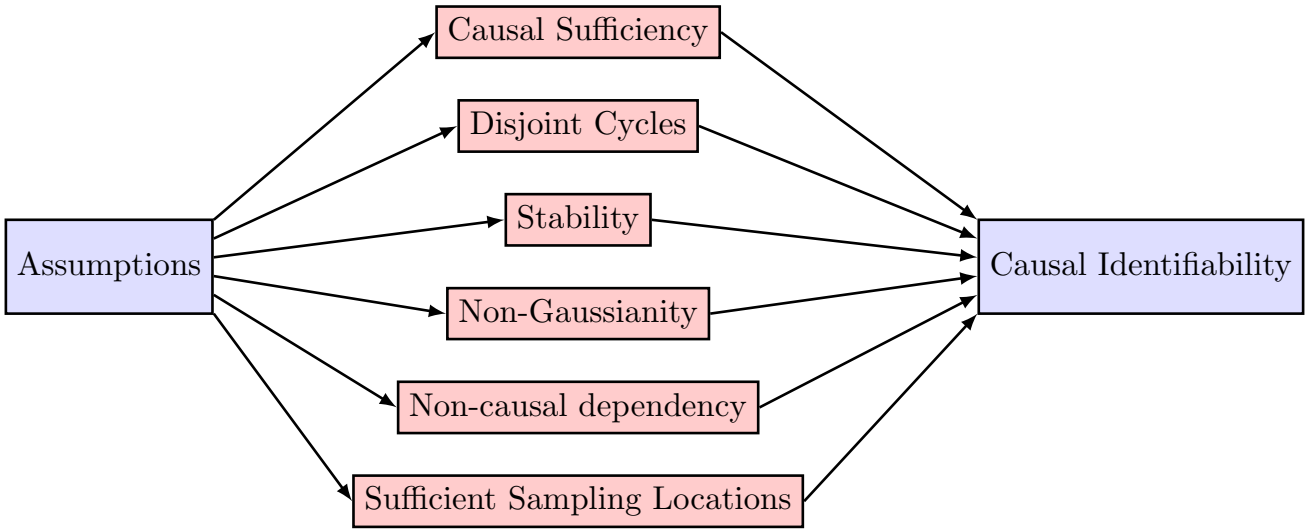
where $\boldsymbol{\Phi}(\boldsymbol{t}) = \text{diag}(\boldsymbol{\Phi}_1(\boldsymbol{t}_1), \ldots, \boldsymbol{\Phi}_p(\boldsymbol{t}_p))$ with $\boldsymbol{\Phi}_j(\boldsymbol{t}_j) = (\phi_{jv}(t_{ju}))_{u,v=1}^{m_j,K_j}$.

Assumption 6 (Sufficient sampling locations)

The basis matrix $\boldsymbol{\Phi}(\boldsymbol{t})$ of size $\sum_{j=1}^{p} m_j \times \sum_{j=1}^{p} K_j$ has a full column rank.

- Implies enough sampling locations over which each random function Y_j is observed.
- Captures all the pertinent causal information that Y_j contains.

Main thorem



Theorem (Causal Identifiability)

Under Assumptions 1-6, $\mathcal{S} = (\mathcal{G}, \mathbb{P})$ is causally identifiable.

Inference - Model parameters

- $\boldsymbol{E} = (E_{j\ell})_{j,\ell=1}^{p}$ denote the adjacency matrix.
- $X_{ju} = Y_j(t_{ju}) + e_{ju}$, $e_{ju} \sim \mathrm{N}(0, \sigma_j)$
- Suppose $Y_j = \sum_{k=1}^{S} \tilde{\alpha}_{jk}\phi_k$.
- We define SEM on first K_j components.

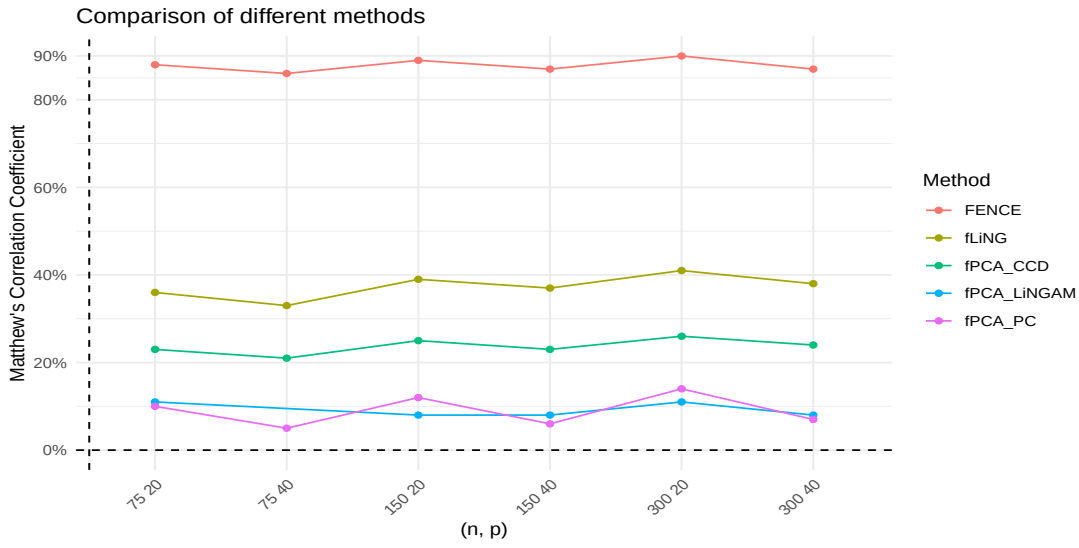
$$\tilde{\boldsymbol{\alpha}} = \boldsymbol{B}\tilde{\boldsymbol{\alpha}} + \tilde{\boldsymbol{\epsilon}}$$

  where $\tilde{\boldsymbol{\alpha}}_j = (\tilde{\alpha}_{j1}, \cdots, \tilde{\alpha}_{jK_j})^\top$ and $\tilde{\boldsymbol{\alpha}} = (\tilde{\boldsymbol{\alpha}}_1, \ldots, \tilde{\boldsymbol{\alpha}}_p$.
- ϕ_k's are useful for restricting each Y_j to D_j. Do not fix them.
- Further expand them with known cubic b-spline basis functions, $\phi_k(\cdot) = \sum_{r=1}^{R} A_{kr} b_r(\cdot)$.
- $\epsilon_{jk} \overset{\text{ind}}{\sim} \sum_{m=1}^{M_{jk}} \pi_{jkm} \mathrm{N}(\mu_{jkm}, \tau_{jkm})$.
- Some very std. priors are taken to formulate the Bayesian procedure.

Simulation study setup

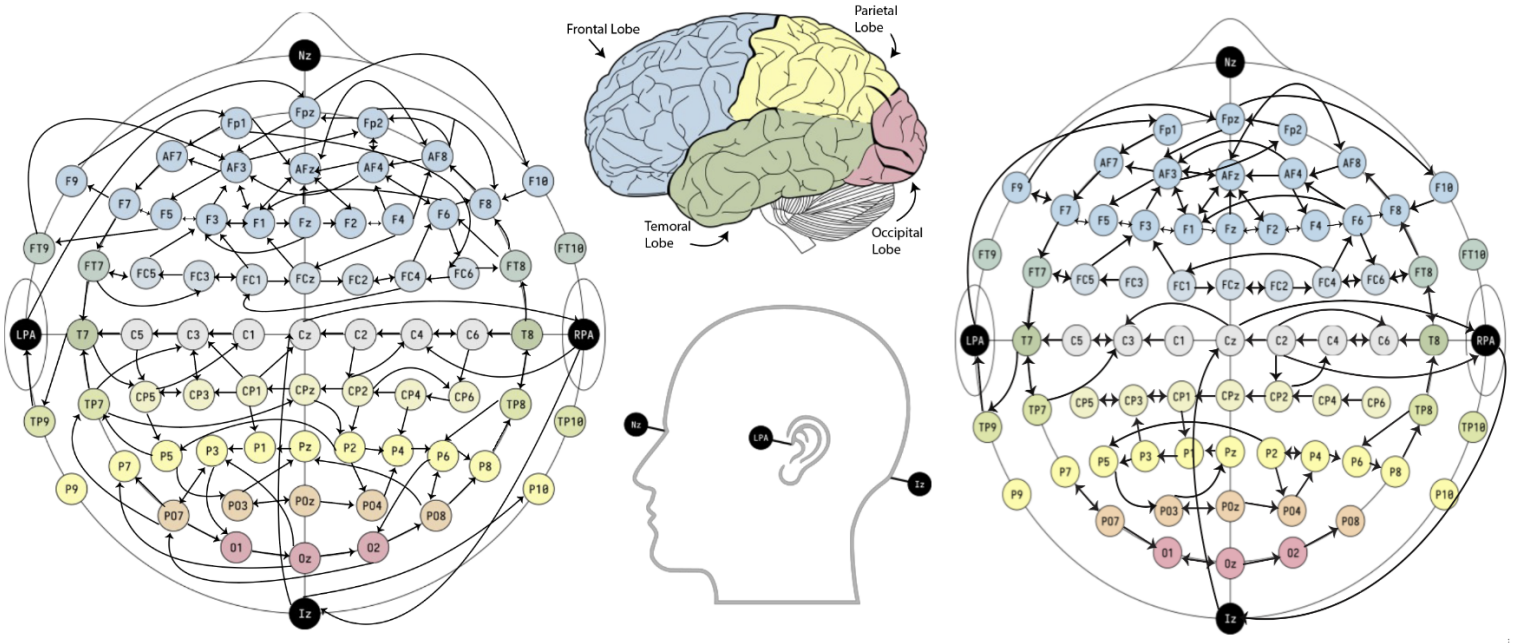
- Sample size $(n) = \{75, 150, 300\}$, number of nodes $(p) = \{20, 40\}$, time grid size $(d) = 125$.
- The causal graph G is estimated by thresholding the posterior probability of inclusion at 0.5.
- Methods compared against:
  1. Functional Bayesian Network (fLiNG).
  2. fPCA-CCD
  3. fPCA-PC
  4. fPCA-LiNGAM
- 2, 3 and 4 are two step procedures:
  – First step involves obtaining the basis coefficients by carrying out fPCA.
  – Second step involves estimating the causal graphs using existing causal discovery methods.

Results



- We used Matthew's correlation coefficient (MCC) to assess the graph recovery performance.

Application - EEG Data



- For both groups (alcoholic and control), brain regions that are spatially closer to each other tend to be more connected.
- Dense connectivity is observed in the frontal region of the brain in both groups, with multiple cycles being formed.
- Alcoholic group has more connectivity across the left parietal and occipital lobes.

Conclusions

- Proposed an operator-based non-recursive linear SEM based framework for functional data in the presence of cycles.
- Introduced the assumption of existence of a lower dimensional causal embedded space that captures all the causal information.
- Proved causal identifiability of our model under certain assumptions.
- Showed applications over an EEG dataset.

Future directions:

- Relaxing the assumption for causal sufficiency.
- Consider non linear SEM framework.

Reference

Roy, S., Wong, R. K. W., & Ni, Y. (2023). Directed Cyclic Graph for Causal Discovery from Multivariate Functional Data. Accepted in NeurIPS, 2023.

Thank you!

Backup slides

Why stability assumption?

- We assume our data are drawn from an equilibrium distribution of a dynamic system involving multivariate functions.
- $\mathbf{Y}(\cdot)[t] = (Y_1(\cdot)[t], \ldots, Y_p(\cdot)[t])^\top$ denote a vector of functions at time point t where the domain of each function $Y_j(\cdot)[t]$ is not necessarily time.
- we have used () to denote the function input/domain and [] to denote the time index of a dynamic system.
- Consider an AR1-type dynamic system of those functions,

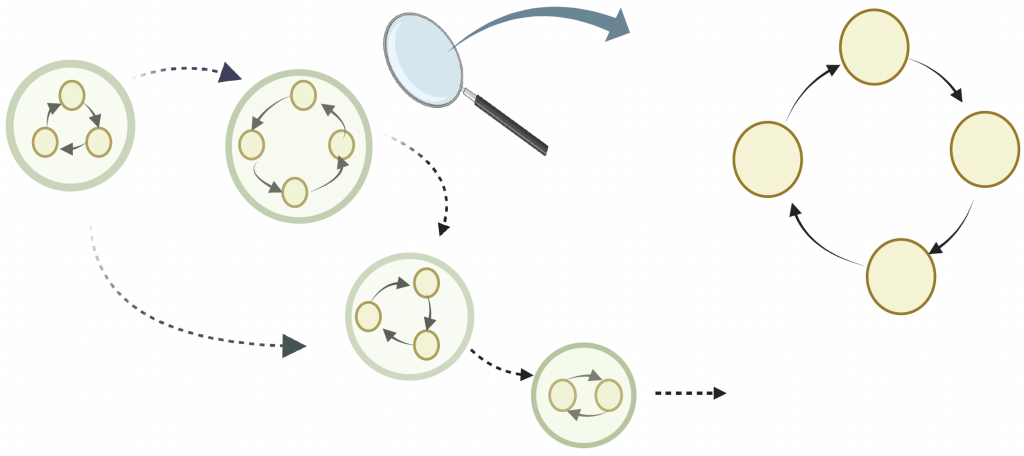
$$\mathbf{Y}(\cdot)[t] = \mathfrak{B}\mathbf{Y}(\cdot)[t-1] + \mathbf{f}(\cdot)$$

- Recursively leads to,

$$\mathbf{Y}(\cdot)[t] = \mathfrak{B}^t\mathbf{Y}(\cdot)[0] + \sum_{s=0}^{t-1} \mathfrak{B}^s\mathbf{f}(\cdot)$$

- $\mathfrak{B}^t$ and $\sum_{s=0}^{t-1} \mathfrak{B}^s$ converge as t approaches infinity.

Proof – A brief sketch pictorially



- In LHS, a hypergraph-like structure emerges when we assume the existence of disjoint cycles.
- While the true graph contains cycles, this hypergraph-like structure is essentially a DAG, thus easier to work with.
- LHS to RHS.

Prior specifications

Prior on spline coefficients $\boldsymbol{A}_k$

$$\boldsymbol{A}_k|\lambda_k \sim \mathrm{N}(\mathbf{0}, \lambda_k^{-1}\boldsymbol{\Omega}^{-})$$

where $\boldsymbol{\Omega}^{-}$ is the pseudoinverse of $\boldsymbol{\Omega} = \int \boldsymbol{b}''(t)[\boldsymbol{b}''(t)]^\top \, dt$. We constrain the regularization parameters $\lambda_1 > \cdots > \lambda_S > 0$ by putting a uniform prior:

$$\begin{aligned}
&\lambda_k \sim \mathrm{Uniform}(L_k, U_k), \ \forall\ k \in [S], \\
&U_1 = 10^8, L_k = \lambda_{k+1} \ \forall\ k \in [S-1], \\
&U_k = \lambda_{k-1} \ \forall\ k \in \{2, \ldots, S\}, L_S = 10^{-8},
\end{aligned}$$

which implies that the smoothness of $\phi_k(\cdot)$ decreases as k gets larger.

Prior specifications

Prior on adjacency matrix $\boldsymbol{E}$

$$E_{j\ell}|\rho \overset{\text{ind}}{\sim} \text{Bernoulli}(\rho)$$
$$\rho \sim \text{Uniform}(0,1)$$

- The marginal distribution of $\boldsymbol{E}$ with ρ integrated out is $\text{Beta}\left(\sum_{j\neq\ell} E_{j\ell} + 1, \sum_{j\neq\ell}(1 - E_{j\ell}) + 1\right)$.
- If $\boldsymbol{E}_0$ denotes the null adj. matrix and $\boldsymbol{E}_1$ denotes the adj. matrix with only one edge, then we can see that $p(\boldsymbol{E}_0)/p(\boldsymbol{E}_1) = p^2 - p$.
- prevents false discoveries and leads to a sparse network by increasing the penalty against additional edges as the dimension p grows.

Prior specifications

Prior on the causal effect matrix $\boldsymbol{B}$

$$\boldsymbol{B}_{j\ell}|E_{j\ell} \sim (1-E_{j\ell})MVN(\boldsymbol{B}_{j\ell};\mathbf{0}, s\gamma\boldsymbol{I}_{K_j}, \boldsymbol{I}_{K_\ell}) + E_{j\ell}MVN(\boldsymbol{B}_{j\ell};\mathbf{0}, \gamma\boldsymbol{I}_{K_j}, \boldsymbol{I}_{K_\ell})$$
$$\gamma \sim \mathrm{IG}(a_\gamma, b_\gamma)$$

with $s = 0.02, a_\gamma = b_\gamma = 1$.

- continuous spike and slab prior.
- When $E_{j\ell} = 0$, $\boldsymbol{B}_{j\ell}$ is negligibly small.

Prior on the noise variances σ_j

$$\sigma_j \sim \mathrm{IG}(a_\sigma, b_\sigma)$$

with $a_\sigma = b_\sigma = 0.01$.

Prior specifications

Prior on the mixture distribution parameters

$$(\pi_{jk1}, \ldots, \pi_{jkM_{jk}}) \sim \text{Dirichlet}(\beta, \ldots, \beta), \qquad \forall\ j \in [p], k \in [S]$$
$$\mu_{jkm} \sim N(a_\mu, b_\mu),\ \tau_{jkm} \sim \text{IG}(a_\tau, b_\tau),\ \forall\ j \in [p], k \in [S], m \in [M_{jk}]$$

We have fixed values for the hyperparameters, $\beta = 1$,
$a_\mu = 0, b_\mu = 100$, $a_\tau = b_\tau = 1$.

Choice of K_j

- Possible to use a prior to learn the number of basis functions jointly with other parameters.
- The functional observations are imputed and arranged into a $(n \times p) \times d$ matrix, where $d = |\cup_{i,j} \mathcal{T}_j^{(i)}|$ represents the size of the union of the measurement grid over all realized random functions.
- Singular value decomposition is performed, and the minimum value of K is selected such that its proportion of variance explained is at least 90%.
- We set $K_j = K$. Note that although K is fixed, the basis functions are adaptively inferred.
- Fix a grid encompassing values $\{1, 2, 3, 4, 5, 6, 7\}$ for K and subsequently selecting the K associated with the lowest WAIC.
- The graph recovery performance remains almost the same.